fall; for which he had little or no care taken of him. He has ever ſince this accident had ſome complaints in his ſide at times, but not conſtantly; nor have they ever been ſo bad, as to prevent his acting in his buſineſs as a ſailor, till within a few weeks before he applied to me.

London, Hatton Garden,
June 28, 1753.

XL. *Extract of a Letter from Mr.* James Dodſon *to Mr.* William Mountaine, *F.R.S.*

May 26, 1753.

Read July 5, 1753.

THE world has, without diſpute, been obliged to the invention of fluxions, for many conciſe methods of calculating the peripheries, areas, and ſolidities, of curvilinear figures; but it muſt be confeſſed, at the ſame time, that the moſt uſeful, even of thoſe, had been computed before, tho' by methods more laborious; and, conſequently, ſince the truth of the principles of fluxions was long diſputed, that art ſeems rather to have received, than to have afforded, any advantage, in thoſe caſes.

Neper and Briggs calculated their ſeveral tables of logarithms, with almoſt inſuperable labour; and Van Ceulen was rendered famous for his approximation to the quadrature of the circle, on account of the acknowleged tedioufneſs of its computation. The methods of computing logarithms were indeed improved, by the aſſiſtance of the properties of the Hyperbola, and the

the ufe of infinite feries, before the great inventer of fluxions was known in the world: but then, the bufinefs of logarithms being purely arithmetical, the Hyperbola was foreign to the fubject; and the nature of infinite feries, tho' well adapted to the purpofe, was at that time but little underftood.

I am fometimes induced to believe, that, if the latter (*i. e.* the nature and ufe of infinite feries) had arrived at any degree of perfection, before the invention of fluxions, moft of the feries, which are given for the above-named kind of calculations, and have been deduced from fluxional proceffes, would have been difcovered without the affiftance of them: and I am of this opinion, becaufe I am certain, that many of them might have fo been: to inftance in both the cafes above quoted; *viz.* the feries, for making logarithms, and for rectifying the circle.

And firft, the terms of one of the moft fimple feries, for expreffing the logarithm of a given number, is compofed of the powers of the excefs of that number, above unity, divided by their refpective indices; of which the firft, third, fifth, *&c.* terms are affirmative, and the fecond, fourth, fixth, *&c.* terms are negative; and the difference between the fums of the affirmative and the negative terms, is the Neperian, hyperbolic, or (as fome call it) the natural logarithm of the given number.

Now a mathematician, who underftands the nature and management of feries (altho' wholly ignorant of fluxions, or what the juftly celebrated Dr. Halley, in his admired inveftigation of this very feries, publifhed in N° 216 of the *Philofophical Tranfactions*, calls *ratiunculæ*, *&c.*) might arrive at the fame conclufion, in the following manner:

Since

Since the logarithm of unity is univerſally determined to be nothing; that of 2, 3, 4, 10, or any other number, conſider'd as a root, is one; that of 4, 9, 16, 100, *&c.* conſider'd as the ſquare of that root, is two; and ſo on; it follows, that (in all caſes) the logarithm of a greater number will exceed that of its leſſer; and each logarithm will have ſome relation to the exceſs of its number, above unity, the number, whoſe logarithm is nothing: the terms of the ſeries, therefore, which will repreſent the logarithm of any number, will conſiſt of the powers of the exceſs of that number, above unity, with ſome, yet unknown, but conſtant coefficients.

That the logarithm of the ſquare of any number is twice the logarithm of its root, is a well-known property of thoſe artificial numbers; and therefore, the doubles of the particular terms of the aſſumed ſeries will conſtitute a ſeries, expreſſing the logarithm of the ſquare of the given number.

But, by the fourth propoſition of the ſecond book of Euclid, the ſquare of any quantity is equal the ſum of the ſquares of its two parts, more a double rectangle of thoſe parts; which, in this caſe (where the given number has been aſſumed, to conſiſt of unity, and an exceſs) will be unity, more twice that exceſs, more the ſquare thereof.

If, therefore, the ſeveral powers of the compound quantity (twice the exceſs of the given number above unity, more the ſquare thereof) be multiplied by the above aſſumed coefficients, and afterwards ranged under each other, according to the powers of the ſaid exceſs; their ſums will again expreſs the logarithm of the ſquare of the given number.

 Now,

Now, ſince the logarithm of the ſquare of the given number may be thus expreſſed by two infinite ſeries, each conſtituted of the exceſs thereof, above unity, and its powers; it follows, that the coefficients of the like powers of that exceſs, in each ſeries, will be equal between themſelves; and, conſequently, the values of the unknown coefficients may be obtained, by ſimple equations; and theſe coefficients will, by the proceſs annexed, appear to be, the reciprocals of the ſeveral indexes of the powers of that exceſs, affected alternately with the ſigns + and —, as was before found, by the quadrature of the Hyperbola, by Dr. Halley in the above-cited *Philoſophical Tranſaction*, and by many who have uſed a fluxional proceſs.

But, there is another logarithmic ſeries equally ſimple with the former, conſiſting of the ſame terms, but all affirmative. This has been demonſtrated to be the logarithm of that fraction, whoſe numerator is unity, and denominator a number, as much leſs than unity, as the former number exceeded it.

Now, if an infinite ſeries be formed from that fraction, by actual diviſion, it will conſiſt of unity, and all the powers of that defect; and if the ſeveral powers of the exceſs of this infinite ſeries above unity, be multiplied by the coefficients above-found, and ranged according to the powers of that defect, their ſums will exhibit the above-deſcribed ſeries for the logarithm of that fraction, as appears by the operation ſubjoin'd.

As to the application of theſe two ſeries, and their ſum, to the finding the logarithms of numbers; the ſame, being copiouſly treated of by Dr. Halley (in the *Philoſophical Tranſaction* before quoted) there is

no

no occasion for the repetition thereof here. *Note*, The above tract of Dr. Halley's is printed with the explication of Sherwin's tables of logarithms (fol. 10) with many examples of the use thereof annexed.

Secondly, The terms of one of the best series, for the rectification of the circle, are composed of the odd powers of the tangent of any arc, not exceeding 45 degrees, severally divided by their respective indexes; of which the first, third, fifth, *&c.* terms are affirmative; and the second, fourth, sixth, *&c.* terms are negative; and the difference, between the sums of the affirmative and negative terms, is the length of that arc, of which the tangent, and its powers, constitute the series.

Now a mathematician, who understands the nature and management of series, altho' wholly ignorant of fluxions, might investigate this series in the following manner:

It has been geometrically demonstrated, that, the radius of a circle being unity, if the double of the tangent of any arc, be divided by the difference between unity, and the square of that tangent, the quotient will be the tangent of twice the arc.

Now if an infinite series be formed by actual division, its terms will consist of the doubles of the odd powers of the tangent, and will be all affirmative; which series will express the length of the tangent of the double of that arc, whose tangent and its powers constitute the same.

If a series, consisting of the tangent and its powers, with unknown coefficients, be assumed (as in the former case) to express the length of the arc; then the length of the double of that arc may be expressed two ways; *viz.* either by multiplying each term of the ſeries aſſumed by the number two; or by finding the powers of the ſeries above deſcribed (which exhibits the length of the tangent of the double arc) multiplying each power by its proper coefficient, ranging the products under each other (according to the powers of the tangent of the ſingle arc) and finding their ſum.

Now, ſince the length of the double arc may be thus expreſſed, by two infinite ſeries, each conſtituted of the tangent of the ſingle arc, and its powers; therefore the coefficients of the like powers of that tangent, in each ſeries, will be equal between themſelves; and conſequently the values of the unknown coefficients may be obtained by ſimple equations.

Laſtly, ſince the ſeries, which gives the length of the tangent of the double arc conſiſts only of the odd powers of the tangent of the ſingle arc, therefore none of the even powers thereof can range therewith: now theſe will not occur in the odd powers of that ſeries; and therefore the ſeries, aſſumed to expreſs the length of the ſingle arc, whoſe double is to be compared with the ſum of the former, muſt conſiſt only of the odd powers of that tangent; and then the ſeries firſt mentioned reſults from the operation, as will appear by examining the ſame, as hereto annexed.

I am thoroughly ſenſible, that, to the learned, who are already maſters of the inveſtigation of theſe ſeries by other methods, this will appear to be an eſſay of more curioſity than uſe; but, with regard to ſtudents, I humbly conceive it will have its advantages; becauſe by thoſe, who are acquainted with the notation of algebra, and the manner of ſolving

ſimple

ſimple equations, the nature of the ſeries, and the operations above uſed, may be eaſily obtained; and they may be hereby enabled to make a ſcientific uſe of logarithms in arithmetical, and of the rectification of the circle, in geometrical calculations; which at preſent cannot be done, till a much greater progreſs is made in mathematical knowlege, without great labour; as well as to examine and correct the printed tables of both ſorts, if dubious, or found to be erroneous.

The operation neceſſary to find the coefficients of a ſeries, which will expreſs the logarithm of a given number.

If the given number be repreſented by $1+n$, then the following ſeries may be aſſumed to repreſent its logarithm:

$n+xn^2+yn^3+zn^4+un^5$, &c.

And $2n+2xn^2+2yn^3+2zn^4+2un^5$, &c. will repreſent the logarithm of the ſquare of that number; *viz.* of $1+2n+nn$.

But, becauſe $2n+nn$ is the exceſs of $1+2n+nn$, above unity, therefore its logarithm will be alſo expreſſed by

$\overline{2n+nn}+x\times\overline{2n+nn}^2+y\times\overline{2n+nn}^3+z\times\overline{2n+nn}^4$, &c.

Now $\overline{2n+nn}^2 = 4nn+4n^3+n^4$

$\overline{2n+nn}^3 = 8n^3+12n^4+6n^5$, &c.

$\overline{2n+nn}^4 = 16n^4+32n^5$, &c.

$\overline{2n+nn}^5 = 32n^5$, &c.

Therefore,

$\overline{2n+nn} = 2n+nn;$

$x \times \overline{2n+nn}^2 = 4xnn + 4xn^3 + xn^4$

$y \times \overline{2n+nn}^3 = 8yn^3 + 12yn^4 + 6yn^5$ &c.

$z \times \overline{2n+nn}^4 = 16zn^4 + 32zn^5$ &c.

$u \times \overline{2n+nn}^5 = 32un^5$ &c.

And the ſum of theſe is equal to the logarithm of the ſquare of $1+n$.

If an equation be formed, of the coefficients of n^2, in each of theſe expreſſions of the logarithm of that ſquare,

Then $2x = 1 + 4x$; whence $-\frac{1}{2} = x$.

And, by proceding in the ſame manner with the coefficients of n^3, n^4, n^5, &c. and ſupplying the places of x, y, z, &c. as they ariſe, by the numbers (ſo found) we ſhall have

$2y = -\frac{4}{2} + 8y$; whence $+\frac{1}{3} = y$;

$2z = -\frac{1}{2} + \frac{12}{3} + 16z$; whence $-\frac{1}{4} = z$;

$2u = \frac{6}{3} - \frac{32}{4} + 32u$; whence $+\frac{1}{5} = u$;

Conſequently, the logarithm of $1+n$ will be expreſſed by $n - \frac{1}{2}n^2 + \frac{1}{3}n^3 - \frac{1}{4}n^4 + \frac{1}{5}n^5$, &c. as above aſſerted.

Again, ſince $\frac{1}{1-n} = 1 + n + n^2 + n^3 + n^4 + n^5$, &c. as appears by the following diviſion:

$$
\begin{array}{l}
1-n\,)\,1 \quad (\,1+n+n^2, \text{ \&c.} \\
\quad\;\; \underline{1-n} \\
\qquad +n \\
\qquad \underline{+n-nn} \\
\qquad\qquad +nn
\end{array}
$$

And, ſince the exceſs of that ſeries, above unity, is the ſeries $n + n^2 + n^3 + n^4$, &c.

Therefore

Therefore the logarithm of $\frac{1}{1-n}$ will consist of the sums of the powers of that series, multiplied by the above-found coefficients $\frac{1}{1}, -\frac{1}{2}, +\frac{1}{3}, -\frac{1}{4}, +\frac{1}{5}$, &c.

$$\text{Now the} \left\{\begin{matrix} 2 \\ 3 \\ 4 \\ 5 \end{matrix}\right\} \text{Power of that Series will be} \left\{\begin{array}{r} n^2 + 2n^3 + 3n^4 + 4n^5, \text{ \&c.} \\ n^3 + 3n^4 + 6n^5, \text{ \&c.} \\ n^4 + 4n^5, \text{ \&c.} \\ n^5, \text{ \&c.} \end{array}\right.$$

And,

$$\left.\begin{matrix} \frac{1}{1} \\ -\frac{1}{2} \\ +\frac{1}{3} \\ -\frac{1}{4} \\ +\frac{1}{5} \end{matrix}\right\} \text{of that} \left\{\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix}\right\} \text{Power will be} \left\{\begin{array}{r} n + n^2 + n^3 + n^4 + n^5, \text{ \&c.} \\ -\frac{1}{2}n^2 - \frac{2}{2}n^3 - \frac{3}{2}n^4 - \frac{4}{2}n^5, \text{ \&c.} \\ +\frac{1}{3}n^3 + \frac{3}{3}n^4 + \frac{6}{3}n^5, \text{ \&c.} \\ -\frac{1}{4}n^4 - \frac{4}{4}n^5, \text{ \&c.} \\ +\frac{1}{5}n^5, \text{ \&c.} \end{array}\right.$$

The sums of which, *viz.* $\overline{n + \frac{1}{2}n^2 + \frac{1}{3}n^3 + \frac{1}{4}n^4 + \frac{1}{5}n^5, \text{ \&c.}}$ will be the logarithm of $\frac{1}{1-n}$, as above affirmed.

The operation necessary, to find the coefficients of a series, which will express the length of the arc of a circle, by the tangent of that arc, and its powers.

Let a represent the length of the arc, and t the length of the tangent thereof; then the tangent of that arc whose length is $2a$, will be $\frac{2t}{1-tt}$; which fraction is equal to the infinite series, $2t + 2t^3 + 2t^5 + 2t^7 + 2t^9$, &c. See the division.

$$
\begin{array}{l}
1-tt\,)\,2t\qquad(\,2t+2t^3+2t^5,\ \&c.\\
\qquad\quad \underline{2t-2t^3}\\
\qquad\qquad +2t^3\\
\qquad\qquad \underline{+2t^3-2t^5}\\
\qquad\qquad\qquad +2t^5
\end{array}
$$

And by performing the neceſſary multiplications, or diviſions, it will alſo appear, that

$$\overline{\frac{2t}{1-tt}}\Big|^3 = 8t^3+24t^5+48t^7+80t^9,\ \&c.$$

$$\overline{\frac{2t}{1-tt}}\Big|^5 = 32t^5+160t^7+480t^9,\ \&c.$$

$$\overline{\frac{2t}{1-tt}}\Big|^7 = 128t^7+896t^9,\ \&c.$$

$$\overline{\frac{2t}{1-tt}}\Big|^9 = 512t^9,\ \&c.$$

Now if we aſſume, for the value of a, the following ſeries, $t+xt^3+yt^5+zt^7+ut^9$, &c.

Then $2t+2xt^3+2yt^5+2zt^7+2ut^9$, &c. $=2a$.

And becauſe $\frac{2t}{1-tt}$ is the tangent of the arc, whoſe length is $2a$, therefore

$$\frac{2t}{1-tt}+x\times\overline{\frac{2t}{1-tt}}\Big|^3+y\times\overline{\frac{2t}{1-tt}}\Big|^5+z\times\overline{\frac{2t}{1-tt}}\Big|^7\ \&c.=2a.$$

Which expreſſion is equivalent to the ſum of the following ſeries; for

2t

$$\frac{2t}{1-tt} = 2t + 2t^3 + 2t^5 + 2t^7 + 2t^9, \text{ \&c.}$$

$$x \times \overline{\frac{2t}{1-tt}}\Big|^3 = 8xt^3 + 24xt^5 + 48xt^7 + 80xt^9, \text{ \&c.}$$

$$y \times \overline{\frac{2t}{1-tt}}\Big| = 32yt^5 + 160yt^7 + 480yt^9, \text{ \&c.}$$

$$z \times \overline{\frac{2t}{1-tt}}\Big| = 128zt^7 + 896zt^9, \text{ \&c.}$$

$$u \times \overline{\frac{2t}{1-tt}}\Big| = 512ut^9, \text{ \&c.}$$

And (by making an eqnation, between $2x$, the coefficient of t^3 in the first-found value of $2a$, and $2+8x$, the sum of the coefficients of t^3 in the latter) $2x = 2 + 8x$; whence $-\frac{1}{3} = x$.

And, by proceding in the same manner, with the coefficients of t^5, t^7, t^9, *&c.* and supplying the places of x, y, z, &c. (as they arise) by the numbers (so found) we shall have

$$2y = 2 - \frac{24}{3} + 32y; \quad \text{whence } y = +\frac{1}{5}.$$

$$2z = 2 - \frac{48}{3} + \frac{160}{5} + 128z; \quad \text{whence } z = -\frac{1}{7}.$$

And $2u = 2 - \frac{80}{3} + \frac{480}{5} - \frac{896}{7} + 512u$; whence $u = +\frac{1}{9}$.

Therefore we may conclude, that $t - \frac{1}{3}t^3 + \frac{1}{5}t^5 - \frac{1}{7}t^7 + \frac{1}{9}t^9$, *&c.* $= a$.

The application of this series to the rectification of the circle is extant in many authors; particularly in Sherwin's tables of logarithms, above-quoted, folio 55.

When the arc is just 45 degrees, then $t = 1$, and the series becomes $\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9}$, *&c.* which converges exceedingly slow; but, by the assistance of a

 method,

method, given in the appendix to M. De Moivre's *Miscellanea Analytica*, it may be transformed to another, converging quicker; which method is applied to this very series, in folio 362 of the *Mathematical Repository*, Vol. I.

XLI. *A Letter from* John Lining, *M. D. of* Charles-Town, South-Carolina, *to the Rev.* Thomas Birch, *D. D. Secr. R. S. concerning the Quantity of Rain fallen there from* January 1738, *to* December 1752.

Rev. Sir, South Carolina, Charles-Town, April 9, 1753.

Read July 8, 1753.

THE favourable reception, which my former papers met with from the Royal Society, encourages me to send you a table of the quantity of rain, which fell in Charles-Town for these 15 years last past; which, if continued for half a century, might be of use, in discovering to us the changes made in a climate, by clearing the land of its woods. Tho' I formerly sent a table of the rain from 1738 to 1745 inclusive, which is publish'd in N° 487 of the *Philosophical Transactions*; yet, as I thought it would be more convenient to bring the whole into one view, I have not only added to this table the rain of those years, but have likewise distinguished the quantity which fell in the several seasons. In this table I continued the old stile to the first of last February, that the mean quantity, in each month, and in the different seasons, might be given exactly.

As